1.5a first-order linear systems

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Recall: Given a kth-order ODE $f(x^{(k)}, x^{(k-1)}, \dots, \dot{x}, x, t) = 0$, we can convert it to a system of k 1st-order ODEs.

Let $x(t+k) + a_{k-1}x(t+k-1) + \cdots + a_1 x(t+1) + a_0 x(t) = b(t)$

Let
$$Y(t) = \begin{bmatrix} Y_1(t) \\ \vdots \\ \vdots \\ Y_k(t) \end{bmatrix} \quad \text{and} \quad \left.\begin{aligned} Y_1(t) &= x(t) \\ Y_2(t) &= x(t+1) = Y_1(t+1) \\ Y_3(t) &= x(t+2) = Y_2(t+1) \\ &\vdots \\ Y_k(t) &= x(t+k-1) = Y_{k-1}(t+1) \\ \text{Then } Y_k(t+1) &+ a_{k-1}Y_k(t) + \cdots + a_1 Y_2(t) + a_0 Y_1(t) = b(t) \end{aligned}\right\} \text{k 1st-order difference equations}$$

Or equivalently, $Y(t+1) = AY(t) + B(t)$, where

$$A = \underbrace{\begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{k-1} \end{pmatrix}}_{\text{companion matrix}} \quad \text{and} \quad B(t) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$$

Given a system $X(t+1) = AX(t) + B(t)$, use Principle of Superposition.

If $X_h(t+1) = AX_h(t)$ and $X_p(t+1) = AX_p(t) + B(t)$, then $X_h + X_p$ is a sol.

Also, $X_h(t) = \sum_{i=1}^{k} c_i X_i(t)$, where $X_i(t)$ are lin. ind. sol. to hom. equation.

Sol. to hom. eq.: $X_h(t+1) = AX_h(t)$.

Ansatz: $X_h(t) = \lambda^t V$, $\lambda \in \mathbb{R}$, $V \in \mathbb{R}^k$

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Then $\lambda^{t+1} V = \lambda^t V$

$\Rightarrow \lambda V = AV$

$\Rightarrow (\lambda, V)$ is an eigenpair of A.

If A has k distinct eigenvalues, then also k lin. ind. eigenvectors, so the sol of the form $\lambda_i^t V_i$ are linearly ind.

Sometimes, will have $t^n \lambda^t V$ if not k distinct eigenvalues.

Note, if eigenvalues are complex, often we will want to write in real form, so we get things like $t^n r^t \sin(\phi t) V$.

Def. 1.9 If $A \in \mathbb{R}^{k \times k}$ has k eigenvalues $\lambda_1, \dots, \lambda_k$, then the spectral radius $\rho(A) = \max\limits_{i \in \{1, \dots, k\}} \{|\lambda_i|\}$.

Thm 1.1 Let $A \in \mathbb{R}^{k \times k}$. Then $\rho(A) < 1$ iff $\lim\limits_{t \to \infty} A^t = 0$.

pf. Recall that any $A \in \mathbb{C}^{k \times k}$ is triangularizable to

$PAP^{-1} = T$, where T is upper triangular with eigenvalues along the diagonal and P is an invertible matrix.

Then $A^t = P^{-1}[PAP^{-1}]^t P = P^{-1} T^t P$.

$\lim\limits_{t \to \infty} T^t = 0$ because the diagonal goes to 0 iff $|\lambda_i| < 1$. $\square$